Nuclear Engineering 282, UC Berkeley

Charged Particle Sources and Beam Technology

Accelerator Physics Fundamentals II
Transverse Beam Dynamics

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Topics

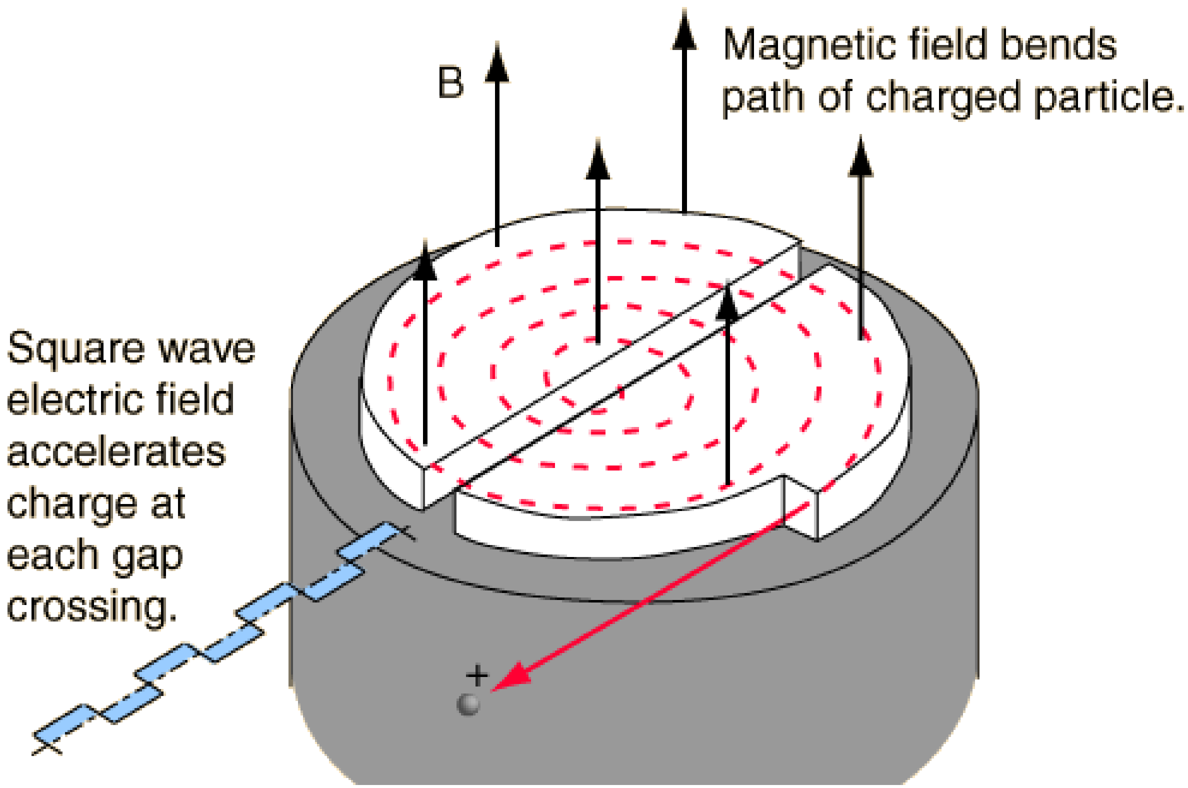
- Transverse Beam Dynamics (single particles)
  - Weak focusing
  - Strong Focusing
    - Matrix formalism – Tracking
    - Hill's Equation: Differential Equation
    - Lattice/Optical Functions
    - Some example lattices
    - Other transverse dynamics concepts
      - Betatron tune
      - Dispersion/Momentum compaction
      - Chromaticity
      - Emittance
      - Resonances
- Summary

Copies of lectures can be found at

http://als.lbl.gov/als_physics/robin/Teaching/UC Nuclear Engineering 282c/

Cyclotron (Livingston and Lawrence 1931)

- ***The first circular accelerator of practical importance based on the principle of repetitive acceleration was the cyclotron, invented by Ernest Orlando Lawrence.***



Cyclotron

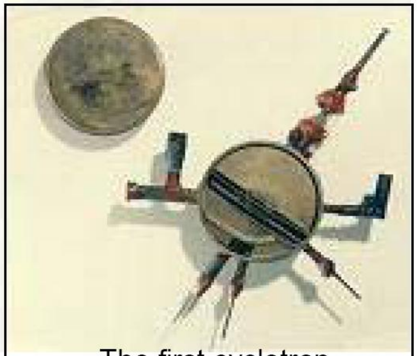
- **In a cyclotron, the charged particles circulate in a strong magnetic field and are accelerated by electric fields in one or more gaps. After having passed a gap, the particles move inside an electrode and are screened from the electric field. When the particles exit from the screened area and enter the next gap, the phase of the time-varying voltage has changed by 180 degrees so that the particles are again accelerated.**
- **Cyclotron condition:**

Centripetal force=Lorentz force=$e\left[\vec{E}+\vec{v}\times\vec{B}\right]$

$$\frac{mv^2}{\rho}=evB$$

$$\rho=\frac{mv}{eB}=\frac{p}{eB}$$

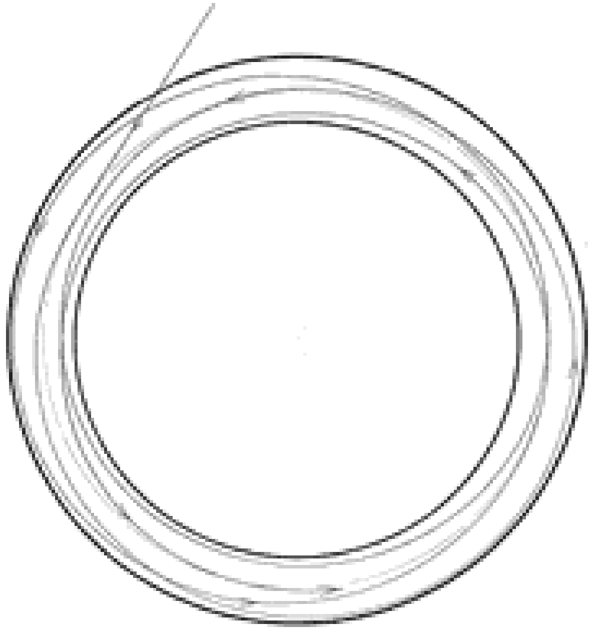
$$f=\frac{v}{2\pi\rho}=\frac{eB}{2\pi m}$$



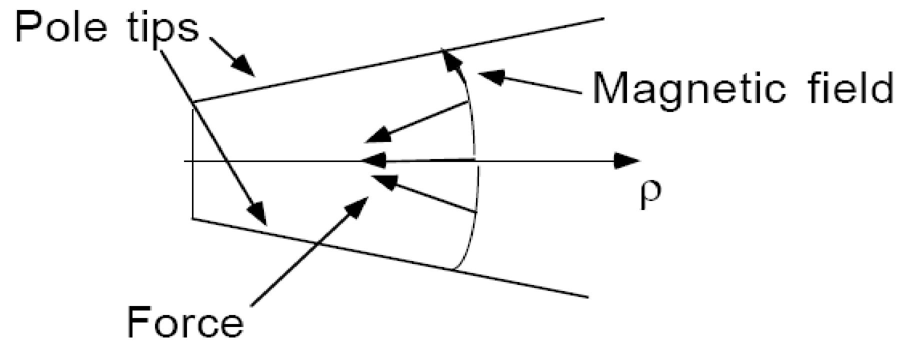
The first cyclotron
4.5” diameter (1929).

- **Only works for non-relativistic particles**

Weak Focusing



Weak focusing accelerator



- In horizontal plane a homogenous dipole magnetic field focuses (intersecting circles)
- Vertically however, trajectories are just diverging.
- Introducing a field gradient provides vertical focusing (and reduces horizontal focusing)
- It also causes particles to get out of sync

with RF faster

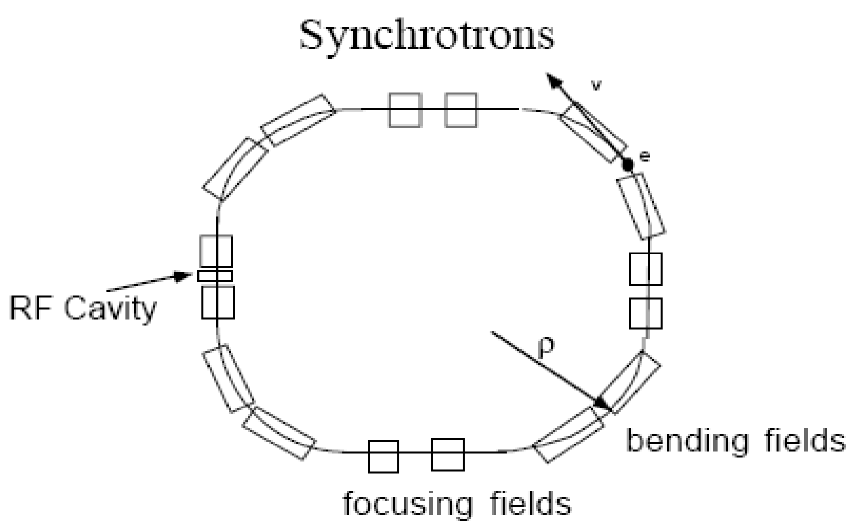
Limitation – Special Relativity

- **Relativistic:** $\vec{p} = \gamma m \vec{v}$
- **Particles in cyclotron become non-synchronous to RF**
- **Options:**
  - **—Synchrocyclotron, i.e. change RF-frequency over time**
    - **No cw/multibunch operation possible, I.e. strong reduction in beam current**
  - **—Isochron-cyclotron:**
    - **Increase field strength radially (causes weak vertical defocusing)**
    - **Use strong focusing principle instead:**
    - **See later slides on strong focusing synchrotrons**

$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$ is positive for a large range of focal lengths and d=> net focusing both radially and vertically

Synchrotron (1945)

- Synchrotrons as well as the linear accelerators (linacs) mentioned before, are important in elementary particle physics research, where highest possible particle energies are needed.
- A synchrotron is a circular accelerator which has one (or a few) electromagnetic resonant cavity to accelerate the particles. A constant orbit is maintained during the acceleration.
  - First ones were weak focusing (very large vacuum chambers and magnets)
  - Later strong focusing.
- Originally ramping/cycling, today often storage rings (many h)



The synchrotron concept seems to have been first proposed in 1943 by the Australian physicist Mark Oliphant.

Weak Focusing Synchrotrons

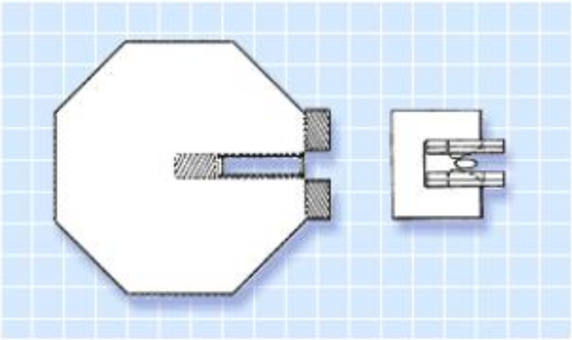
Figure 3.3. Cross section of weak focusing circular accelerator.

Cosmotron

- As with cyclotrons, the first synchrotrons were of the so called weak-focusing type. The first synchrotron of this type was the Cosmotron at the Brookhaven National Laboratory, Long Island. It started operation in 1952 and provided protons with energies up to 3 GeV.
- In the early 1960s, the world’s highest energy weak-focusing synchrotron, the 12.5 GeV Zero Gradient Synchrotron (ZGS) started its operation at the Argonne National Laboratory near Chicago, USA.
- The Dubna synchrotron, the largest of them all with a radius of 28 meters and with a weight of the magnet iron of 36,000 tons.
- The Bevatron at Berkeley Lab was a weak focusing synchrotron it operated from 1954 to the early 1990s and the antiproton was found there in 1955 – Nobel Prize for Segre and Chamberlain in 1959.

Strong Focusing (Synchrotrons)

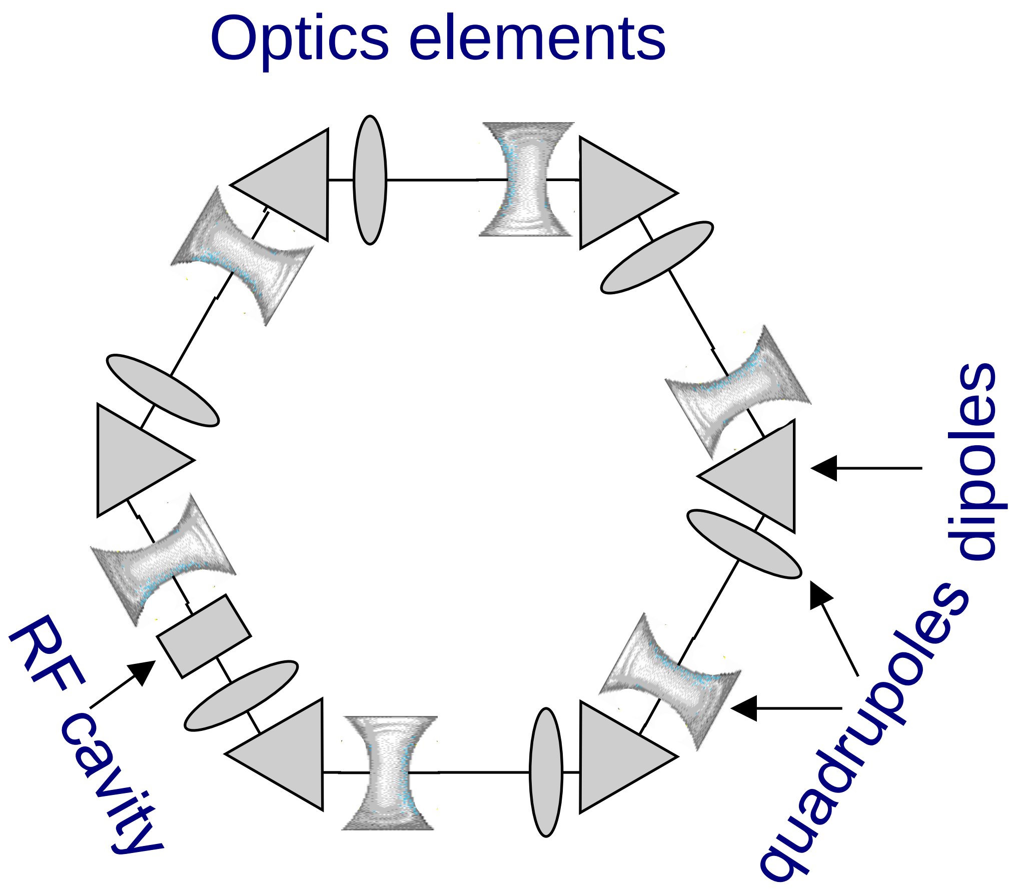
- In 1952 Courant, Livingston and Snyder proposed a scheme for strong focusing of a circulating particle beam so that its size can be made smaller than that in a weak focusing synchrotron.
- Later rediscovered that Christofilos had proposed concept already many years earlier.
- Bending magnets are made to have alternating magnetic field gradients; after a magnet with ar axial field component decreasing with increasing radius follows one with a componen increasing with increasing radius and so on.
- Thanks to the strong focusing, the magnet apertures can be made smaller and therefore much less iron is needed.
- The first alternating-gradient synchrotron accelerated electrons to 1.5 GeV. It was built at Cornell University, Ithaca, N.Y. and was completed in 1954.

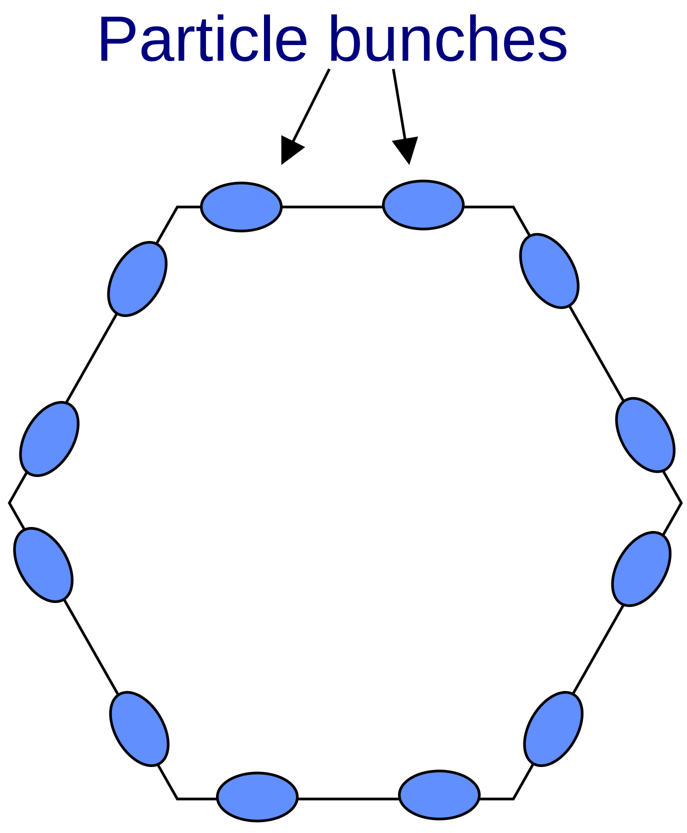


Size comparison between the Cosmotron's weak-focusing magnet (L) and the AGS alternating gradient focusing magnets

Synchrotrons / Storage Rings

In a particle storage rings, charged particles circulate around the ring in bunches for a large number of turns.





Equations of Motion in a Storage Ring

The motion of each charged particle is determined by the electric and magnetic forces that it encounters as it orbits the ring:

- **Lorentz Force**

$$F = ma = e(E + v \times B),$$

m **is the relativistic mass of the particle,**

e **is the charge of the particle,**

v **is the velocity of the particle,**

a **is the acceleration of the particle,**

E **is the electric field and,**

B **is the magnetic field.**

Typical Magnet Types

There are several magnet types that are used in storage rings:

Dipoles → used for guiding

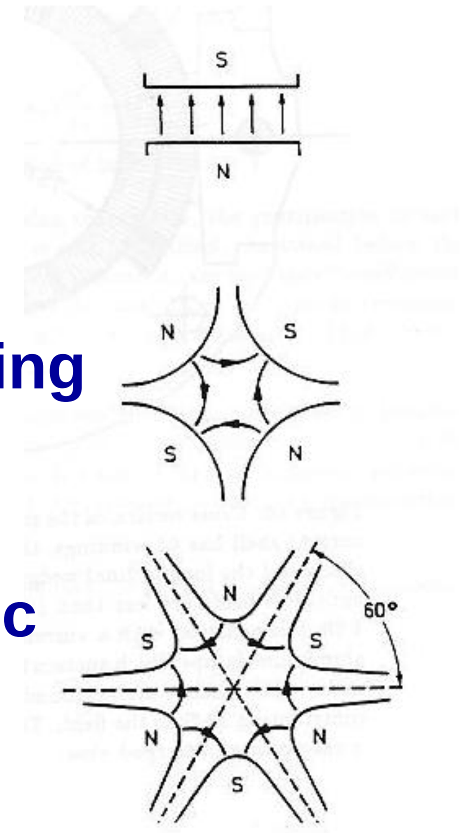
$$B_x = 0$$
$$B_y = B_o$$

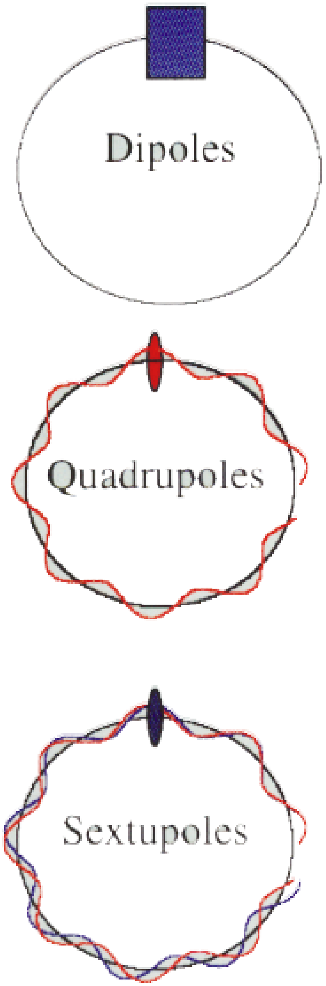
Quadrupoles → used for focussing

$$B_x = Ky$$
$$B_y = -Kx$$

Sextupoles → used for chromatic correction

$$B_x = 2Sxy$$
$$B_y = S(x^2 - y^2)$$





Practical Magnet Examples at the ALS

Dipoles

Quadrupoles

Sextupoles

Two approaches

There are two approaches to introduce the motion of particles in a storage ring

1. **The traditional way in which one begins with Hill's equation, defines beta functions and dispersion, and how they are generated and propagate, …**

2. **The way that our computer models actually do it**

I will begin with the second way and then go back to the first.

Equations of motion

Begin with equations of motion → Lorentz force

↓

Change dependent variable from time to longitudinal position

↓

Integrate particle trajectory around the ring and find the closed orbit

↓

Generate a map around the closed orbit

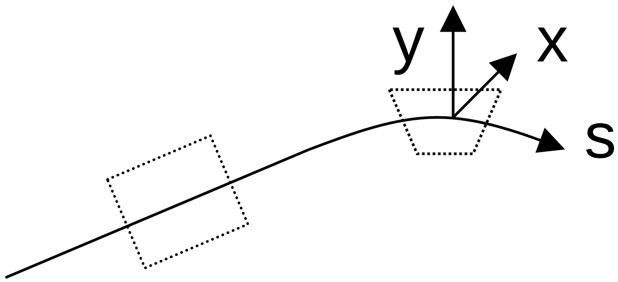
↓

Analyze and track the map around the ring

Coordinate System

Change dependent variable from time to longitudinal position, *s*

Coordinate system used to describe the motion is usually locally Cartesian or cylindrical



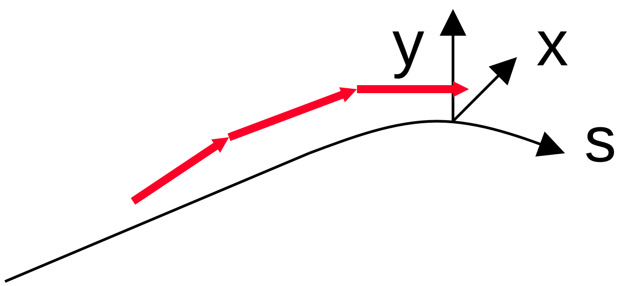
Typically the coordinate system chosen is the one that allows the easiest field representation

Integrate

Integrate through the elements

Use the following coordinates*

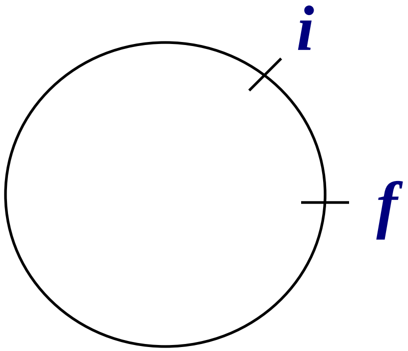
$$x, \quad x' = \frac{dx}{ds}, \quad y, \quad y' = \frac{dy}{ds}, \quad \delta = \frac{\Delta p}{p_0}, \tau = \frac{\Delta L}{L}$$



Note sometimes one uses canonical momentum rather than x' and y'

Transfer Matrix

One can write the linear transformation between one point in the storage ring (i) to another point (f) as



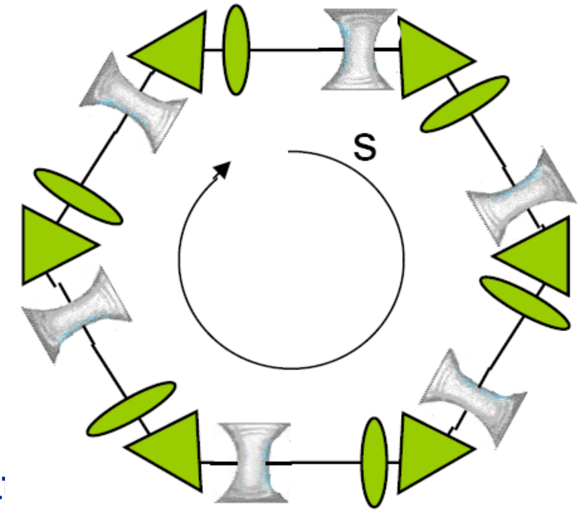
$$\begin{pmatrix} x \\ x' \end{pmatrix}_f = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

this is for the case of uncoupled horizontal motion. One can extend this to 4x4 or 6x6 cases.

Piecewise constant magnetic fields

- **General** transfer matrix **from s_0 to s**

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0)\begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$



- **Note that**

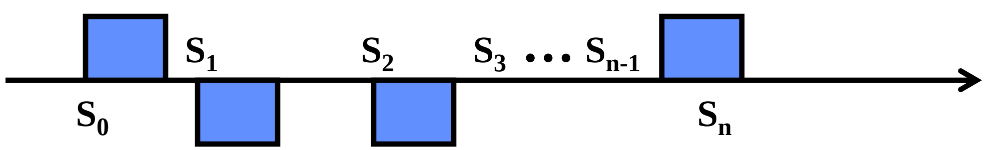
$$\det(\mathcal{M}(s|s_0)) = C(s|s_0)S'(s|s_0) - S(s|s_0)C'(s|s_0) = 1$$

which is always true for conservative systems

- **Note also that** $\mathcal{M}(s_0|s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}$

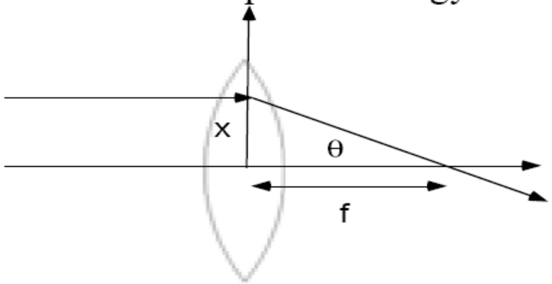
- **The accelerator can be build by a series of matrix multiplica**

$$\mathcal{M}(s_n|s_0) = \underbrace{\mathcal{M}(s_n|s_{n-1})\ldots\underbrace{\mathcal{M}(s_3|s_2)\cdot\underbrace{\mathcal{M}(s_2|s_1)\cdot\underbrace{\mathcal{M}(s_1|s0)}_{\text{from } \mathbf{s_0} \text{ to } \mathbf{s_1}}}_{\text{from } \mathbf{s_0} \text{ to } \mathbf{s_2}}}_{\text{from } \mathbf{s_0} \text{ to } \mathbf{s_3}}}_{\text{from } \mathbf{s_0} \text{ to } \mathbf{s_n}}$$

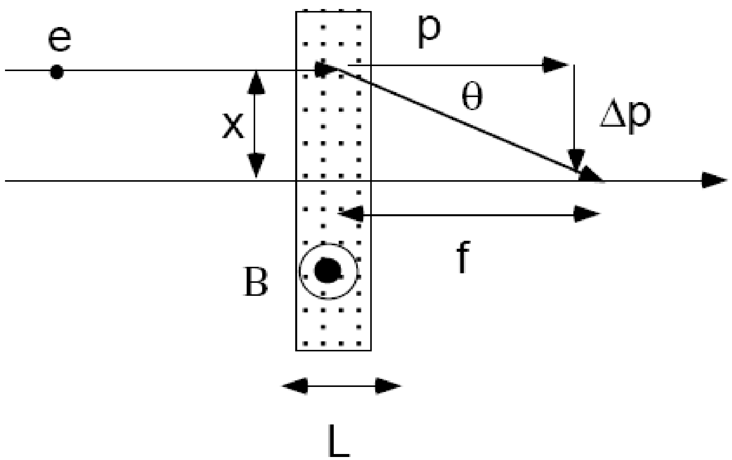


Magnetic lenses: Quadrupoles

Magnetic focusing fields:
Optical analogy: Thin lens, focal length f

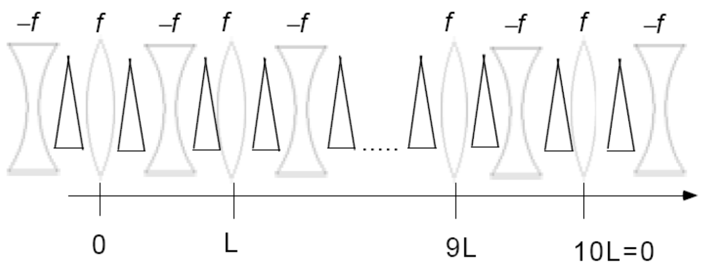


$$\boxed{\theta = \frac{x}{f}}$$



- **Thin lens representation**

FODO cell

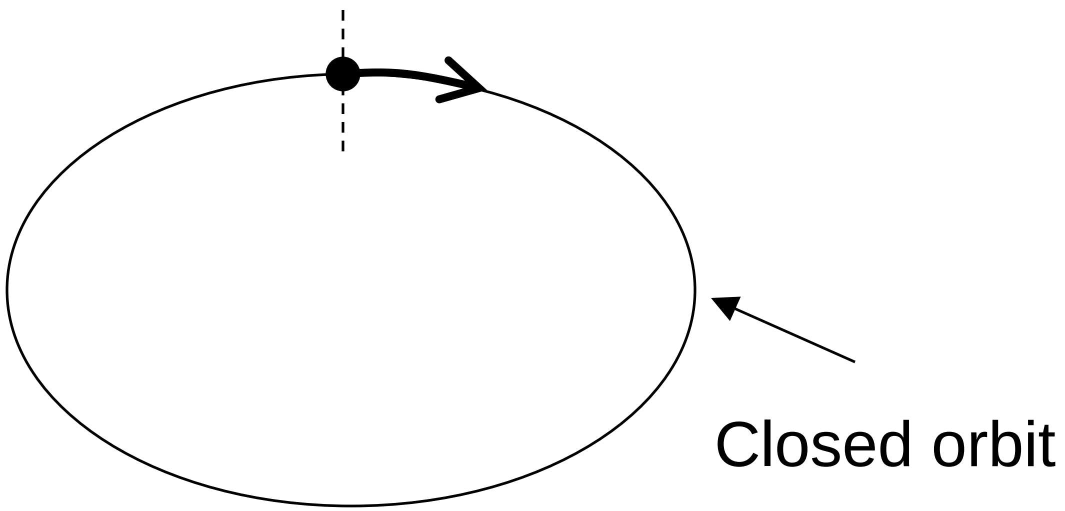


$$\begin{pmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{f} & 1 \end{pmatrix} \quad \begin{pmatrix} x(s,s_0) \\ x'(s,s_0) \\ y(s,s_0) \\ y'(s,s_0) \\ \delta_l(s,s_0) \\ \delta \end{pmatrix}$$

Drift: Thin lens:

Find the Closed Orbit

A closed orbit is defined as an orbit on which a particle circulates around the ring arriving with the same position and momentum that it began.



In every working story ring there exists at least one closed orbit.

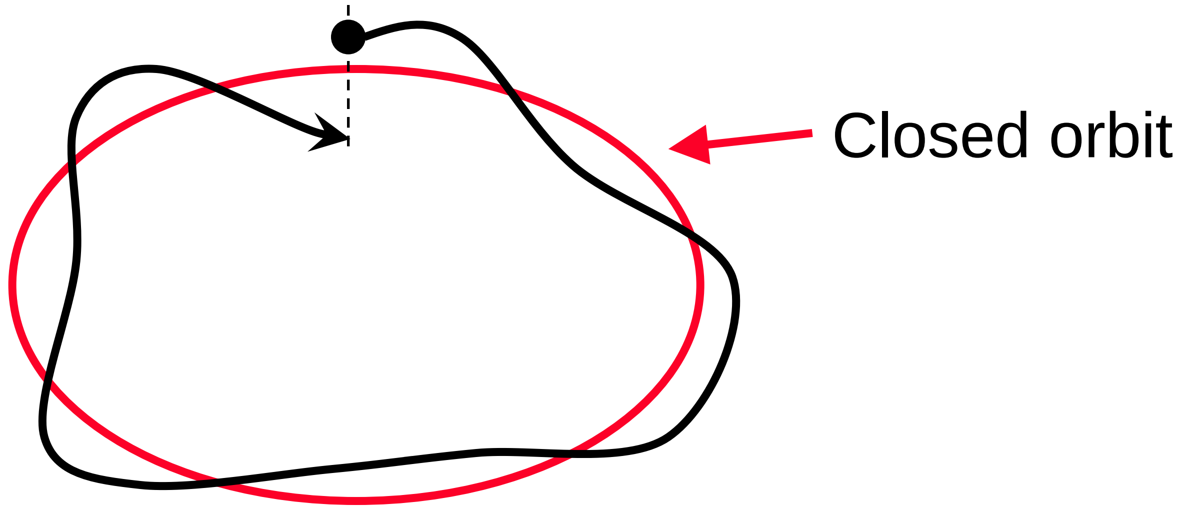
Generate a one-turn Map Around the Closed Orbit

A one-turn map, *R*, maps a set of initial coordinates of a particle to the final coordinates, one-turn later.

$$x_f = x_i + \frac{dx_f}{dx_i}\left(x_i - x_{i,co}\right) + \frac{dx_f}{dx'_i}\left(x'_i - x'_{i,co}\right) + \ldots$$

$$x'_f = x'_i + \frac{dx'_f}{dx_i}\left(x_i - x_{i,co}\right) + \frac{dx'_f}{dx'_i}\left(x'_i - x'_{i,co}\right) + \ldots$$

The map can be calculated by taking orbits that have a slight deviation from the closed orbit and tracking them around the ring.



Computation of beta-functions and tunes

The one turn matrix (the first order term of the map) can be written

$$R_{one-turn} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\varphi + \alpha\sin\varphi & \beta\sin\varphi \\ -\gamma\sin\phi & \cos\varphi - \alpha\sin\varphi \end{pmatrix}$$

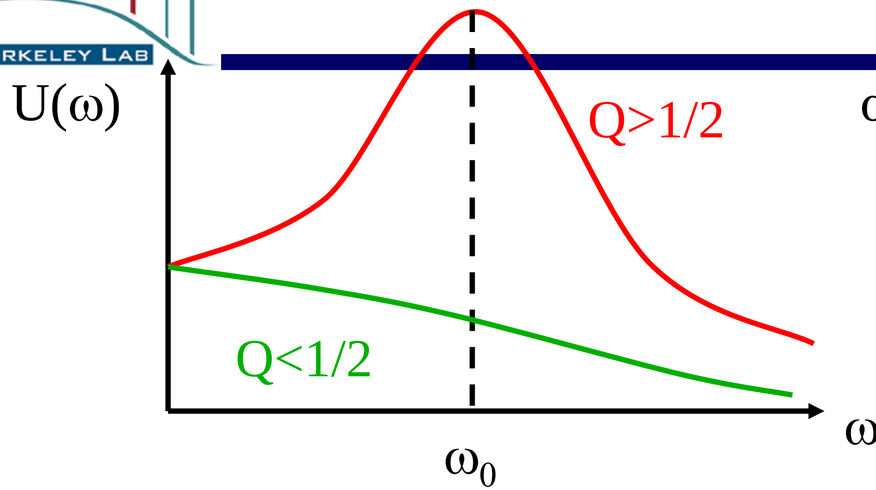
Where α, β, γ are called the Twiss parameters

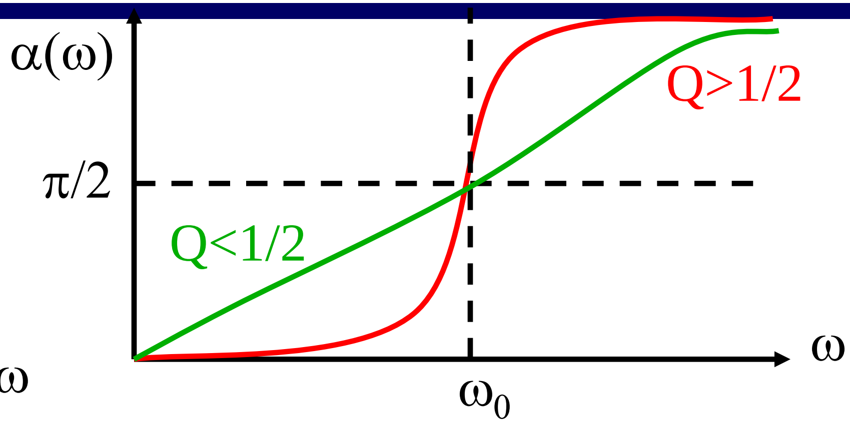
$$\alpha = -\frac{\beta'}{2}, \qquad \gamma = \frac{1+\alpha^2}{\beta}$$

and the betatron tune, $\nu = \phi/(2*\pi)$

For long term stability ϕ is real →

$|TR(R)| = |2\cos\phi| < 2$



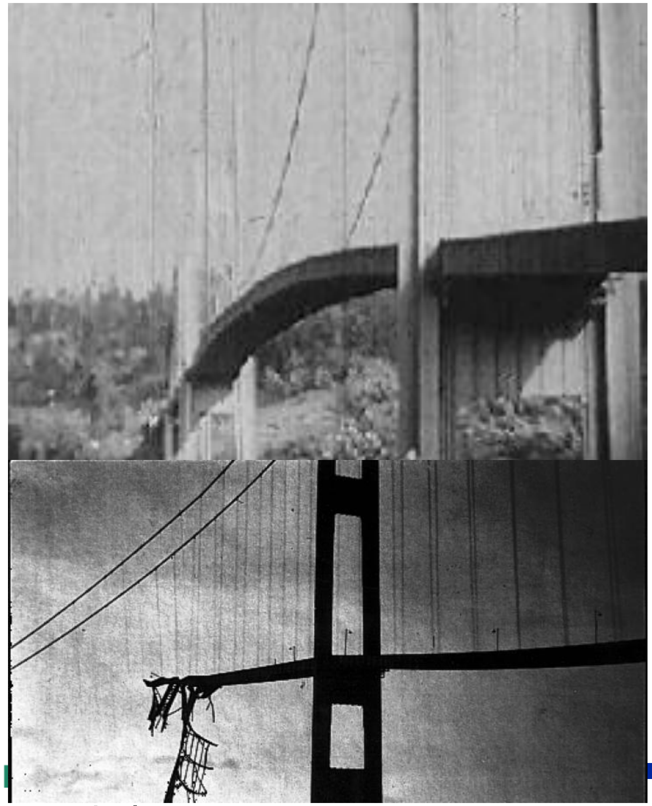


$$U(\omega) = \frac{U(0)}{\sqrt{(1-(\frac{\omega}{\omega_0})^2)^2 + (\frac{\omega}{Q\omega_0})^2}}$$

- Without or with weak damping a resonance condition occurs for $\omega = \omega_0$
- Infamous example:

Tacoma Narrow bridge 1940

strong wind excited motion at the Eigenfrequencies



First approach – traditional one

This approach provides some insights but is limited

Begin with on-energy no coupling case. The beam is transversely focused by quadrupole magnets. The horizontal linear equation of motion is

$$\frac{d^2x}{ds^2} = -k(s)x,$$

where $k = \frac{B_T}{(B\rho)a}$, **with**

B_T **being the pole tip field**

a **the pole-tip radius, and**

$B\rho$**[T-m]** $\approx 3.356\,p$**[GeV/c]**

Hills equation

The solution can be parameterized by a psuedo-harmonic oscillation of the form

$$x_\beta(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\varphi(s)+\varphi_0)$$

$$x_\beta^{'}(s) = -\sqrt{\varepsilon}\frac{\alpha}{\sqrt{\beta(s)}}\cos(\varphi(s)+\varphi_0) - \frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\sin(\varphi(s)+\varphi_0)$$

where $\beta(s)$ is the beta function,

$\alpha(s)$ is the alpha function,

$\varphi_{x,y}(s)$ is the betatron phase, and

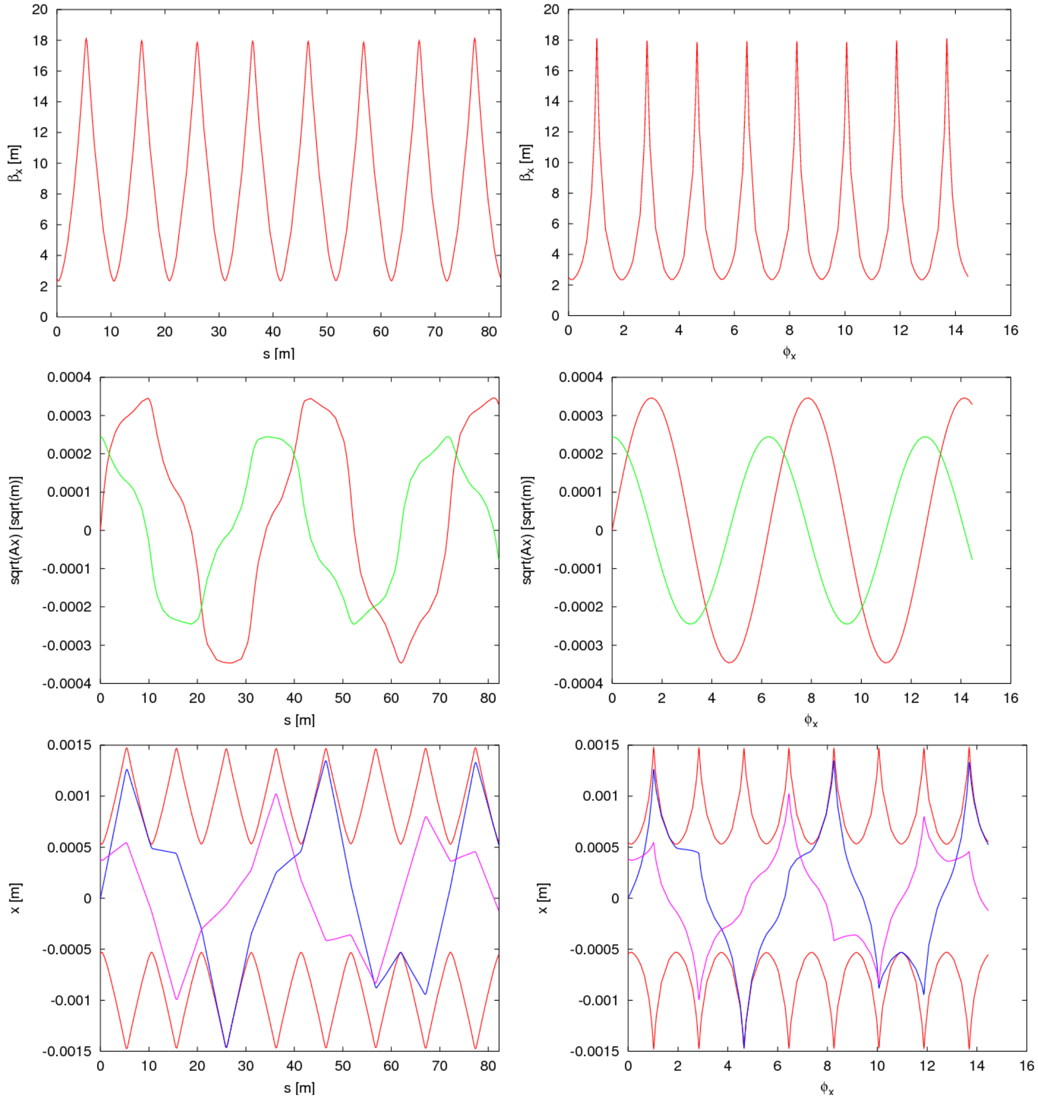
ε is an action variable

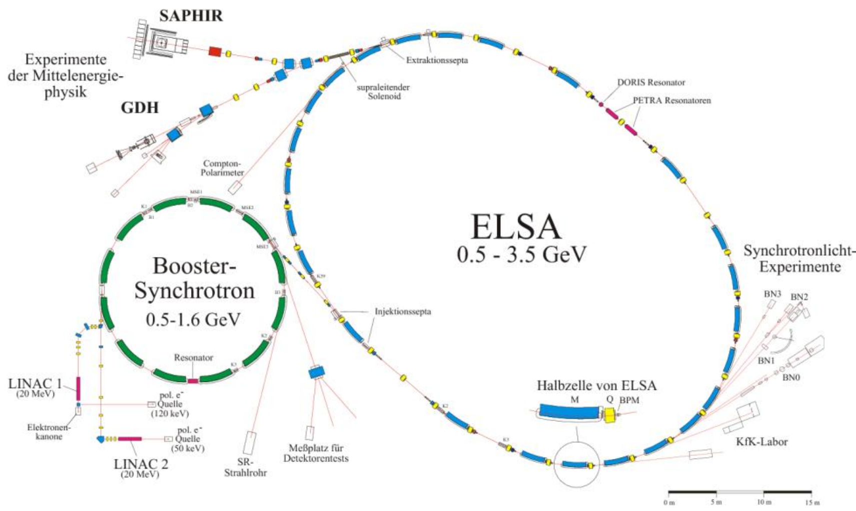
$$\alpha = -\frac{\beta^{'}}{2},$$

$$\varphi = \int_0^s \frac{ds}{\beta}$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

Example of Twiss parameters and trajectories



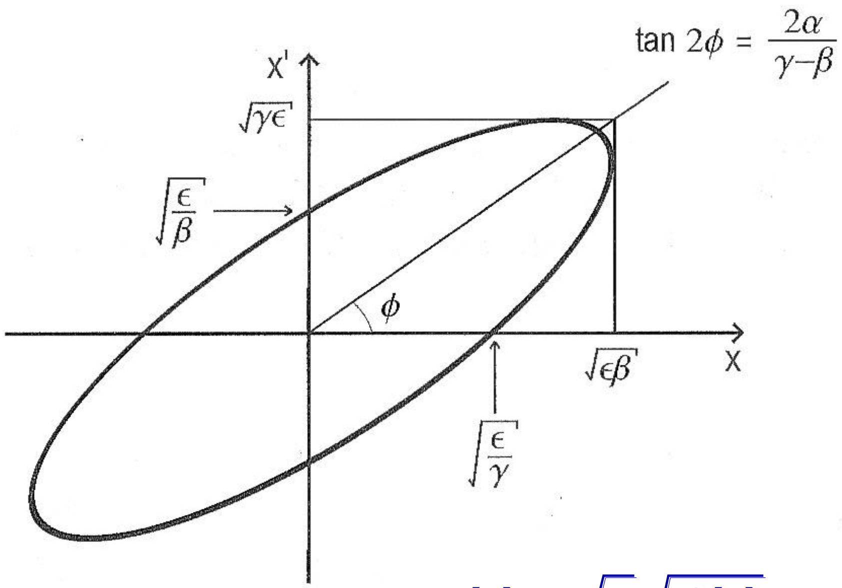


ELSA (Electron Stretcher and Accelerator) in Bonn is an example of a relatively simple FODO lattice

- Beta Function highly periodic
- Trajectories in real space are piecewise straight (with deflections at quadrupoles)
- If one transforms with beta functions and phase advance, they start to look like harmonic functions (sine/cosine)

Beam Ellipse

In an linear uncoupled machine the turn-by-turn positions and angles of the particle motion will lie on an ellipse



Area of the ellipse, ε:

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

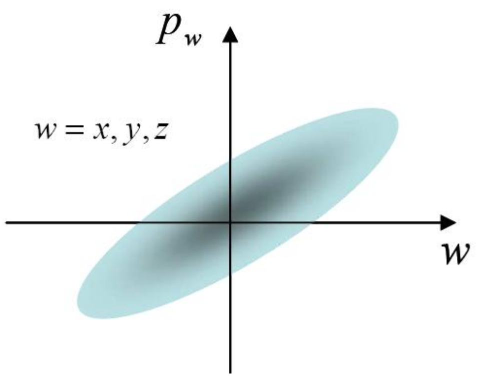
$$x_\beta(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\varphi(s)+\varphi_0)$$

$$x'_\beta(s) = -\sqrt{\varepsilon}\frac{\alpha}{\sqrt{\beta(s)}}\cos(\varphi(s)+\varphi_0) - \frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\sin(\varphi(s)+\varphi_0)$$

Emittance Definition

- Consider the decoupled case and use the {*w*, *w*} plane where *w* can be either *x* or *y*:
  - — The emittance is the phase space area occupied by the system of particles, divided by π

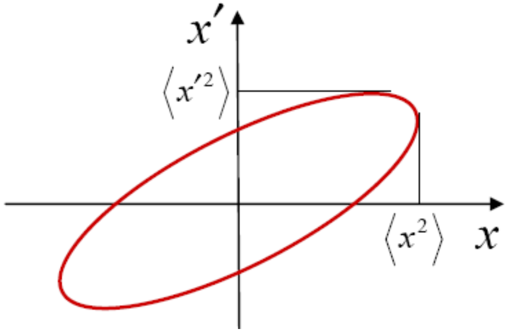
$$\boxed{\varepsilon_w = \frac{A_{ww'}}{\pi} \qquad w = x, y}$$



- *x'* and *y'* are conjugate to *x* and *y* when $B_z = 0$ and in absence of acceleration. In this case, we can immediately apply the Liouville theorem:
  - — For such a system the emittance is an invariant of the motion.
- This specific case is very common in accelerators:
  - — For most of the elements in a beam transferline, such as dipoles, quadrupoles, sextupoles, …, the above conditions apply and the emittance is conserved.

Emittance Definition/Statistical

- **Emittance defined as the phase space area occupied by an ensemble of particles**
- **Example: In the transverse coordinates it is the product of the size (cross section) and the divergence of a beam (at beam waists).**
- **Emittance can be defined as a statistical quantity (beam is composed of finite number of particles)**



$$\varepsilon_{geometric,rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

$$\langle x^2 \rangle = \frac{\sum_{n=1}^{N} x_n^2}{N} \cong \frac{\int x^2 f_{2D}(x,x')\,dx\,dx'}{\int f_{2D}(x,x')\,dx\,dx'}$$

This is equivalent to associate to the real beam an *equivalent or phase ellipse* in the phase space with area $\pi\varepsilon_{rms}$ and equation:

$$\frac{\langle x'^2 \rangle}{\varepsilon_{rms}} x^2 + \frac{\langle x^2 \rangle}{\varepsilon_{rms}} x'^2 - 2\frac{\langle x\,x' \rangle}{\varepsilon_{rms}} x\,x' = \varepsilon_{rms}$$

Transport of the beam ellipse

Beam ellipse matrix

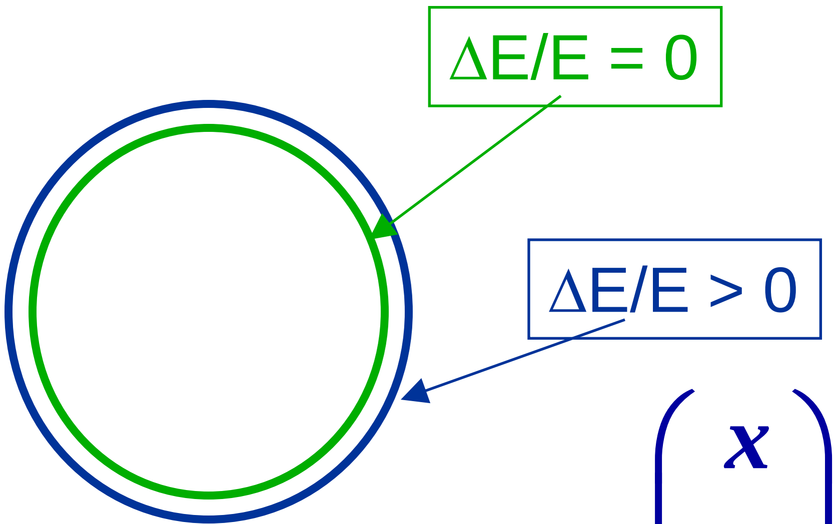
$$\sum\nolimits_{beam}^{x} = \varepsilon_x \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

Transformation of the beam ellipse matrix

$$\sum\nolimits_{beam,f}^{x} = R_{x,i-f} \sum\nolimits_{beam,i}^{x} R^{T}_{x,i-f}$$

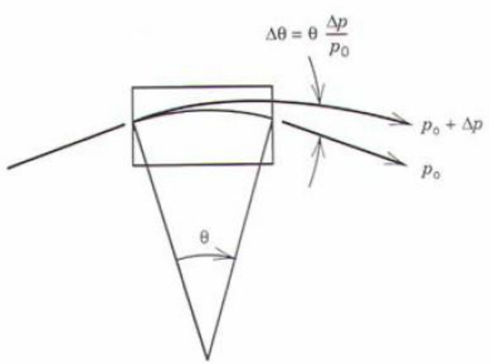
Dispersion

Dispersion, *D*, is the change in closed orbit as a function of energy



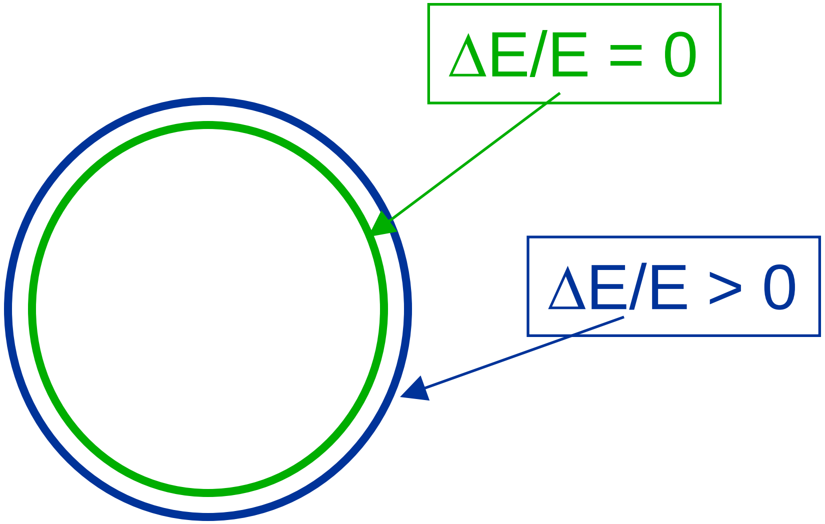
$$x = D_x \frac{\Delta E}{E}$$

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_f = \begin{pmatrix} C & S & D_x \\ C' & S' & D_x' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_i$$



Momentum Compaction

Momentum compaction, α, is the change in the closed orbit length as a function of energy.

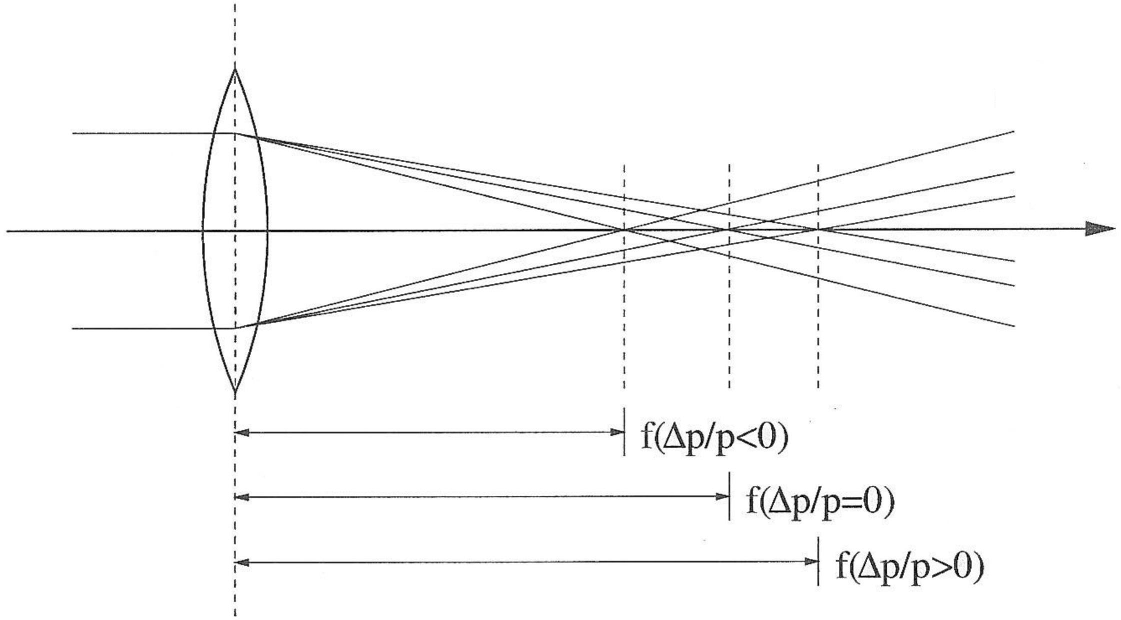


$$\frac{\Delta L}{L} = \alpha \frac{\Delta E}{E}$$

$$\alpha = \int_0^{L_0} \frac{D_x}{\rho} ds$$

Chromatic Aberration

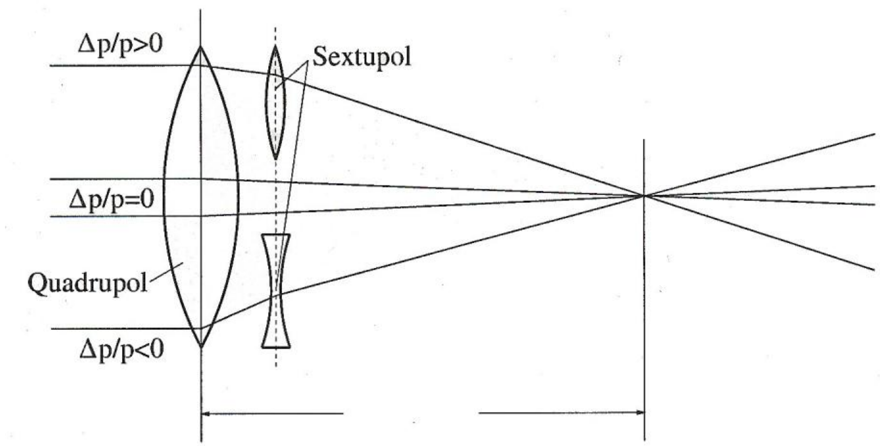
Focal length of the lens is dependent upon energy



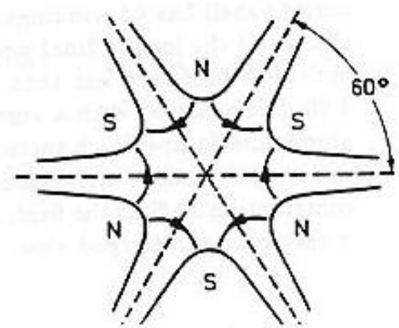
Larger energy particles have longer focal lengths

Chromatic Aberration Correction

By including dispersion and sextupoles it is possible to compensate (to first order) for chromatic aberrations



The sextupole gives a position dependent Quadrupole



$$B_x = 2Sxy$$

$$B_y = S(x^2 - y^2)$$

Chromatic Aberration Correction

Chromaticity, ν', is the change in the tune with energy

$$\nu' = \frac{d\nu}{d\delta}$$

Sextupoles can change the chromaticity

$$\Delta\nu_x{}' = \tfrac{1}{2\pi}\left(\Delta S \beta_x D_x\right)$$

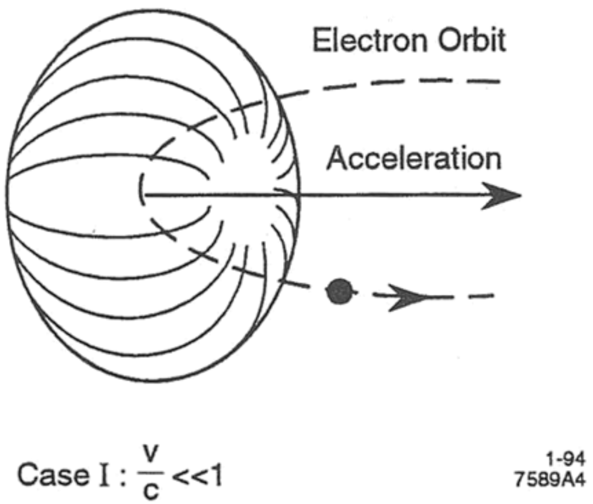
$$\Delta\nu_y{}' = -\tfrac{1}{2\pi}\left(\Delta S \beta_y D_x\right)$$

where

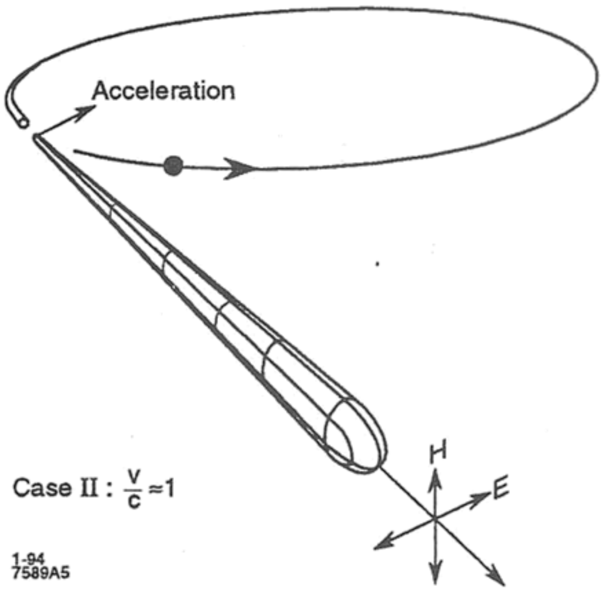
$$\Delta S = \left(\partial^2 B_y \middle/ \partial x^2 \right) \cdot \text{length} \middle/ \left(2B\rho\right)$$

Synchrotron Radiation

- Radiated power increases at higher velocities
- Radiation becomes more focused at higher velocities



At low electron velocity (non-relativistic case) the radiation is emitted in a non-directional pattern



When the electron velocity approaches the velocity of light, the emission pattern is folded sharply forward. Also the radiated power goes up dramatically

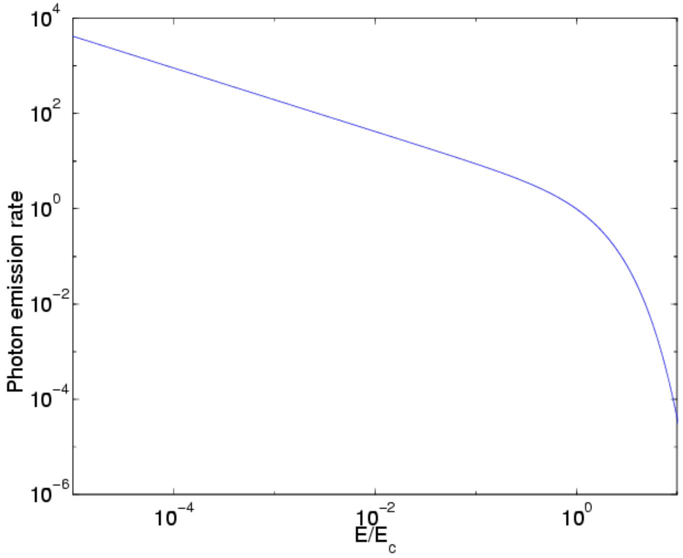
Radiation

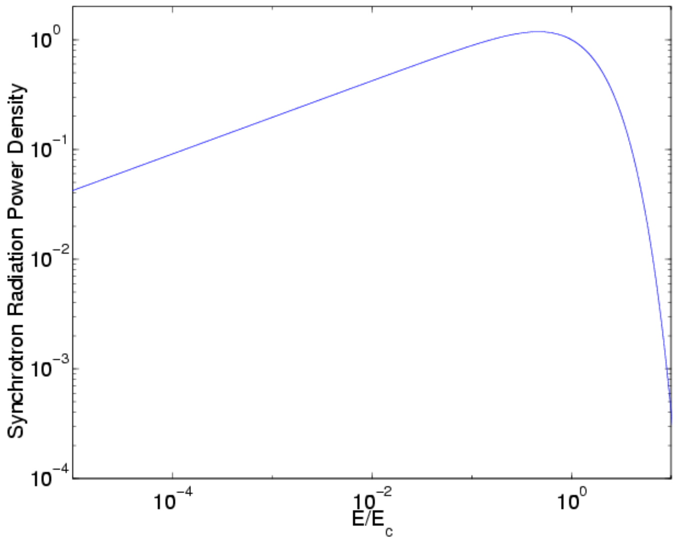
The power emitted by a particle is

$$P_{SR} = \frac{2}{3}\alpha\hbar c^2 \frac{\gamma^4}{\rho^2}$$

and the energy loss in one turn is

$$U_0 = \frac{4\pi}{3}\alpha\hbar c \frac{\gamma^4}{\rho^2}$$





Radiation damping

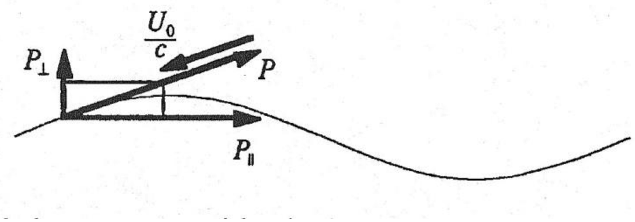
Energy damping:

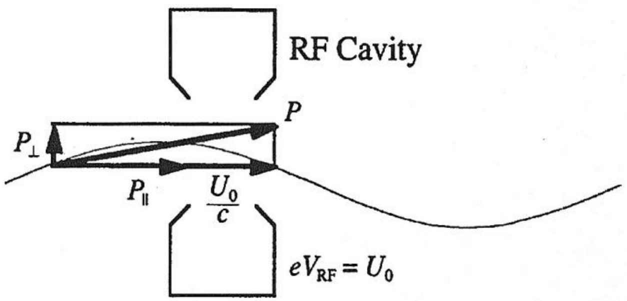
Larger energy particles lose more energy

$$P_{SR} = \frac{2}{3}\alpha\hbar c^2 \frac{\gamma^4}{\rho^2}$$

Transverse damping:

Energy loss is in the direction of motion while the restoration in the s direction





Quantum excitation

The synchrotron radiation emitted as photons, the typical photon energy is

$$u_c = \hbar\omega_c = \frac{3}{2}\hbar c \frac{\gamma^3}{\rho}$$

The number of photons emitted is

$$N = \frac{4}{9}\alpha c \frac{\gamma}{\rho}$$

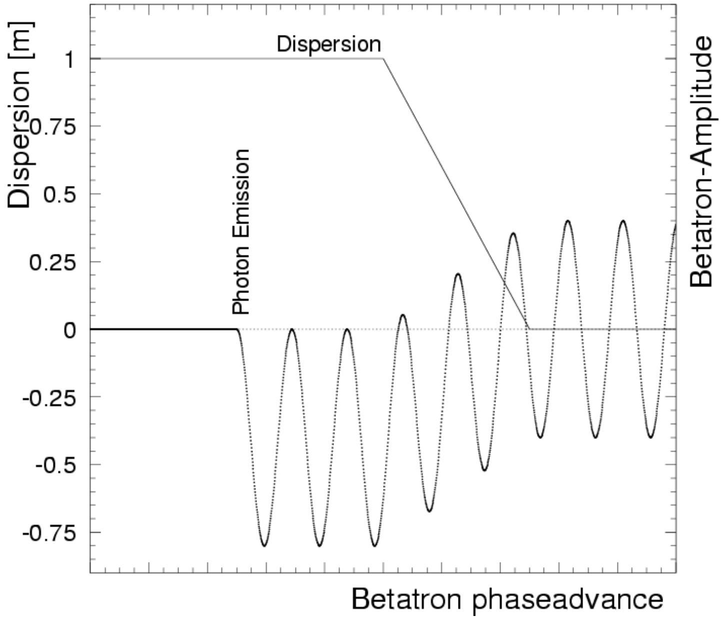
With a statistical uncertainty of $\sqrt{N}$

The equilibrium energy spread and bunch length is

$$\left(\frac{\sigma_e}{E}\right)^2 = 1.468 \cdot 10^{-6} \frac{E^2}{J_\varepsilon \rho} \quad \textbf{and} \quad \sigma_L = \frac{\alpha R}{f_0}\sigma_e$$

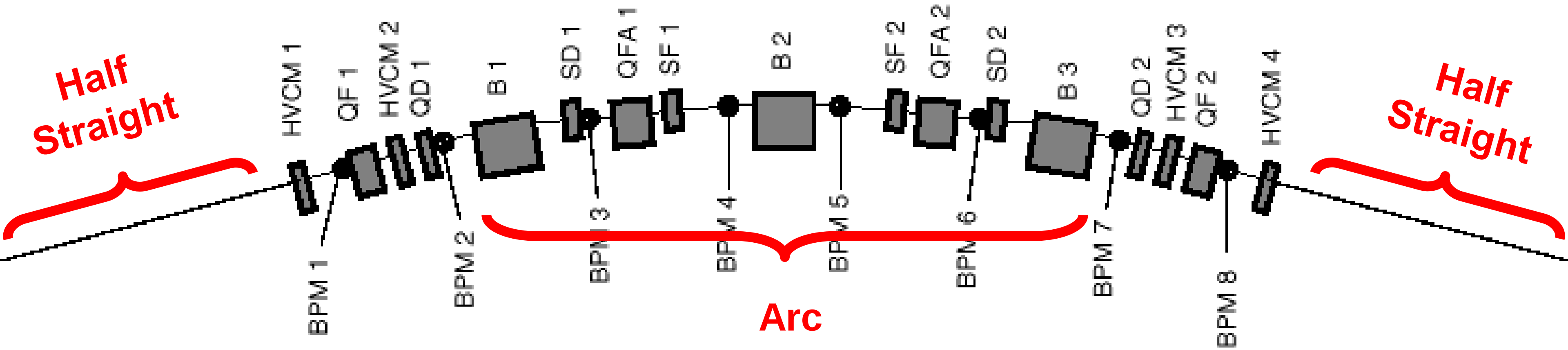
Emittance and beam size

Particles change their energy in a region of dispersion undergoes increase transverse oscillations. This balanced by damping gives the equilibrium emittances.

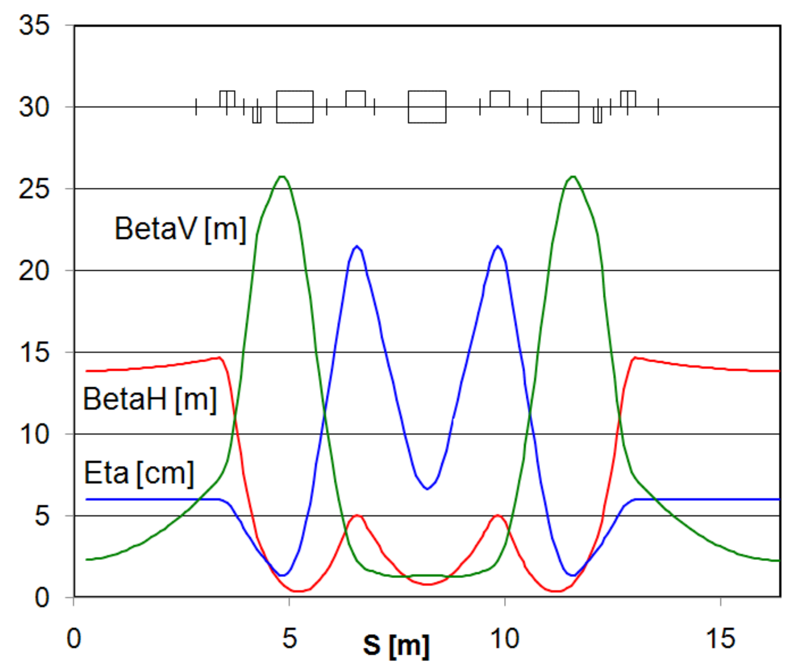


The beam size is $$\sigma_x = \sqrt{\beta_x \varepsilon + \left(D_x \frac{\sigma_e}{E} \right)^2}$$

Other Lattice Example: ALS



- The ALS is an example of a low emittance lattice
- It is a so called triple bend achromat and minimizes the dispersion at the location where synchrotron radiation is emitted (dipoles)
- Those achromat lattices were the major advance in 3rd generation light sources



$$\text{Emittance} \propto \int_{Bend} \frac{\gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2}{\rho^3} ds$$

Summary

- Introduced many concepts of linear beam dynamics
- Matrix (transverse) beam transport approach is helpful to calculate simple problems by hand
- Computer codes use an extension of this approach (nonlinear integrators for individual elements, symplectic, …)
- Find closed orbit – generate map around closed orbit – lattice functions
- Historic and text book approach of hill's equation provides some insight, but is not widely used by computer codes